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BELLMAN'S INCLUSIONS AND EXCESSIVE MEASURES

BY

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Abstract. The paper is concerned with Bellman's inclusions for the value function of the optimal stopping for a Markov process X on a complete separable metric space E. The author investigates a connection between seemingly unrelated objects: excessive measures, differential inclusions and optimal stopping. Conditions are given under which an evolutionary Bellman inclusion has a strong or weak solution in the Hilbert space $L^2(E, \mu)$, where μ is an excessive measure for X. The solution is identified with the value function of a stopping problem. The stationary Bellman inclusion is treated as well. Specific examples of diffusions with jumps and infinite-dimensional diffusions are discussed. Excessivity of the measure μ plays an essential role in the development. The results are then applied to pricing American options both in finite and infinite dimensions recently investigated by Zhang [32], Mastroeni and Matzeu [20], [21], and Gątarek and Musiela [11].

1. INTRODUCTION

Let (E, ϱ) be a metric, complete and separable space equipped with a σ-field of its Borel subsets $\mathscr{E} = \mathscr{B}(E)$ and $P_t(x, \Gamma)$, $t \geqslant 0$, $x \in E$, $\Gamma \in \mathscr{E}$, a transition function of a Markov process X. Assume that $X(t, x)$, $t \geqslant 0$, $x \in E$, is the process X starting from x, defined on a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ with filtration $(\mathscr{F}_t)$. Let α, φ and ψ be real functions on E. For an arbitrary $(\mathscr{F}_t)$-stopping time τ and $s \in [0, +\infty)$ define functionals

$$(1)\quad \mathscr{J}_s(\tau, x) = \boldsymbol{E}\Big(\exp\Big\{\int_0^\tau \alpha\big(X(\sigma, x)\big)\,d\sigma\Big\}\big[\varphi\big(X(\tau, x)\big)\chi_{\tau<s} + \psi\big(X(\tau, x)\big)\chi_{\tau=s}\big]\Big),$$

and let V be the corresponding value function:

$$(2)\qquad V(s, x) = \sup_{\tau \leqslant s} \mathscr{J}_s(\tau, x).$$

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A heuristic, dynamic programming argument leads to the following Bellman's inclusion:

$$\frac{\partial V}{\partial t}(t, x) \in \mathscr{A} V(t, x) + \alpha(x) V(t, x) - \partial I_{\mathscr{K}_\varphi}(V(t, x)), \tag{3}$$

$$V(0, x) = \psi(x), \quad x \in E, \ t \geqslant 0,$$

for the function V. The inclusion is considered in a Hilbert space $\mathscr{H} = L^2(E, \mu)$, where μ is an ω-excessive measure, $\omega \geqslant 0$, for the transition function (P_t). A locally finite measure μ on E is *ω-excessive* for (P_t) (see [13]) if

$$P_t^* \mu(\Gamma) \leqslant e^{\omega t} \mu(\Gamma) \quad \text{for all } t \geqslant 0, \ \Gamma \in \mathscr{E}, \tag{4}$$

where the measure $P_t^* \mu$ is given by

$$P_t^* \mu(\Gamma) = \int_E P_t(x, \Gamma) \mu(dx).$$

In particular, invariant measures for (P_t) are ω-excessive for arbitrary $\omega \geqslant 0$. The 0-excessive measures are called shortly *excessive*. A measure μ is *locally finite* if there exists an increasing sequence of open sets (U_n) such that

$$\mu(U_n) < +\infty, \quad n \in N \text{ and } \bigcup_{n=1}^{+\infty} U_n = E.$$

It turns out that excessive measures are natural weights for the equation (3), see Section 2. Moreover, for arbitrary finite measure ν and $\omega > 0$, the measure μ:

$$\mu = \int_0^{+\infty} P_t^* \nu \, dt,$$

is ω-excessive.

In (3) the symbol $\mathscr{A}$ stands for a proper version of the characteristic operator of the Markov process, $\partial I_{\mathscr{K}_\varphi}$ is the subgradient of the indicator function $I_{\mathscr{K}_\varphi}$ of the set

$$\mathscr{K}_\varphi = \{\psi \in \mathscr{H} : \psi \geqslant \varphi\}.$$

The Bellman inclusion (3) is usually written in a different way and interpreted as an evolutionary variational inequality; see [3]. One needs then however a bilinear form, defined on a subspace on $\mathscr{H}$, which determines the operator $\mathscr{A}$. In the case of general state space E the construction of the bilinear form can be done only in special cases (see [19]) and in the present paper we do not assume its existence.

Rather complete, analytic and probabilistic theories of optimal stopping are available for diffusion processes on open subsets of Euclidean spaces R^d or for Markov processes on locally compact spaces; see [3], [9] and [16]. For financial applications, see [15] and [24]. However, only few papers are dealing with analytic questions for more general spaces and more general processes. Diffusions with jumps are the object of recent papers [32], [20] and [21]. In [7] a class of infinite-dimensional Bellman's stationary inclusions on a Hilbert space was studied. The case of an infinite-dimensional Ornstein–Uhlenbeck process is investigated in a recent paper [11]. More general infinite-dimensional processes are treated by viscosity methods in [12]. Processes on Banach spaces E are studied in [2] under the condition that the transition function defines a C_0-*semigroup* on the space $C_b(E)$. It is a strong condition rarely satisfied even if $E = R^1$.

The initial motivation of the present study was to extend an existing analytic theory of optimal stopping to *infinite-dimensional spaces*, where a natural equivalent of the Lebesgue measure does not exist. We believe however that the approach to Bellman's inclusion based on the concept of excessive measures simplifies several classical proofs and allows to treat situations not covered by existing theories. We derive and extend some of the recent results on optimal stopping of diffusions with jumps (see [32], [20], [21]) and of infinite--dimensional diffusions (see [11], [12]). We prove the existence of strong solutions to (3) in the case of general state space and regular data and identify the weak solutions as value functions.

It is known that *excessive functions* play a fundamental role in the theory of optimal stopping (see e.g. [25]). We hope that the present paper demonstrates that *excessive measures* are useful in the analytic characterization of the corresponding value function. We also believe that a close connection between seemingly unrelated objects: excessive measures, differential inclusions and optimal stopping is of independent interest.

The paper is concerned with existence of solutions and their continuity only. This is a first step to resolve more practical questions: how regular are the solutions and how can they be approximated? Thus the situation is similar to that with viscosity solutions (see Lions [18]). Additional work should be done to find effective ways of calculating the solutions.

The paper is organized as follows. In Section 2 we establish the existence of strong solution to (3) provided functions α, φ and ψ are regular enough. A similar existence result is obtained also for a stationary Bellman's inclusion:

$$0 \in \mathscr{A} V(x) + \alpha(x) V(x) - \partial I_{\mathscr{K}_\varphi}(V(x)) \tag{5}$$

on the value function V corresponding to the problem of maximizing the functional:

$$\mathscr{J}_\infty(\tau, x) = E\big(\exp\{\int_0^\tau \alpha(X(\sigma, x))\, d\sigma\}\, \varphi(X(\tau, x))\, \chi_{\tau < +\infty}\big) \tag{6}$$

with respect to all stopping times τ. We use here classical results on maximal monotone operators and also the excessivity of the measure μ. It is shown in Section 3 that if data φ and ψ are only continuous but satisfy some growth conditions the function V given by (2) is a weak solution to (3). To prove the convergence of an approximating sequence, results on maximal monotone operators are used as well as the penalization technique due to Bensoussan and Lions [3]. Section 4 is devoted to financial applications. In fact, the problem of stopping optimally an infinite-dimensional forward process or, in the financial language, the problem of pricing interest rate derivatives (see [11], [12]), was a starting point for the present paper.

Probabilistic theory of optimal stopping on metric complete separable spaces has been recently discussed in paper [29], to which we will often refer. The present paper is a rewritten version of the preprints [28] and [30]. For proofs that some specific measures are ω-excessive, both in finite and in infinite dimensions, we refer to [30] and [31].

2. STRONG SOLUTIONS OF BELLMAN'S INCLUSIONS

By $B_b(E)$ and $C_b(E)$ we denote the spaces of bounded Borel and bounded continuous functions on E, respectively. In this section we start with two semigroups P_t and R_t of linear operators acting on $B_b(E)$ of the form:

$$P_t\varphi(x) = \int_E P_t(x, dy)\varphi(y), \qquad R_t\varphi(x) = \int_E R_t(x, dy)\varphi(y), \qquad \varphi \in B_b(E), \ x \in H.$$

We will need the following assumption:

(A.1) (i) *For each* $t \geqslant 0$, $x \in H$, $P_t(x, \cdot)$, $R_t(x, \cdot)$ *are nonnegative measures such that for a constant* a:

$$P_t(x, E) = 1, \qquad R_t(x, \Gamma) \leqslant e^{at} P_t(x, \Gamma), \qquad \Gamma \in \mathscr{B}(E).$$

(ii) *For arbitrary* $\varphi \in C_b(E)$, *functions* $P_t\varphi(x)$ *and* $R_t\varphi(x)$, $t \geqslant 0$, $x \in E$, *are continuous.*

In the next section the following semigroups will be defined by probabilistic formulae:

$$P_t\varphi(x) = E\big(\varphi(X(t, x))\big), \qquad t \geqslant 0, \ x \in E, \ \varphi \in B_b(E),$$

$$R_t\varphi(x) = E\Big(\exp\Big\{\int_0^t \alpha(X(s, x))\Big\}\varphi(X(t, x))\Big), \qquad t \geqslant 0, \ x \in E, \ \varphi \in B_b(E),$$

where α is a function on E, bounded from above by the number a. We will require also that:

(A.2) *A locally finite measure* μ *on* E *is* ω-*excessive for the transition function* P_t.

The assumptions (A.1) and (A.2) are satisfied for all examples of interest. We have the following basic theorem the proof of which can be found in [30].

THEOREM 1. *Under the assumptions* (A.1) *and* (A.2) *the semigroup* (R_t) *has a unique extension to a* C_0*-semigroup of positive operators onto* $\mathscr{H} = L^2(E, \mu)$, *such that*

$$|R_t \psi|_{\mathscr{H}} \leqslant \exp\{(\omega/2+a)t\} |\psi|_{\mathscr{H}}, \quad t \geqslant 0, \ \psi \in \mathscr{H}. \tag{7}$$

The extended semigroup will also be denoted by (R_t).

Let $\mathscr{L}$ be the infinitesimal generator of (R_t). For arbitrary $\varphi \in \mathscr{H}$ denote by $\mathscr{K}_\varphi$ the following closed and convex subset in $\mathscr{H}$:

$$\mathscr{K}_\varphi = \{\psi \in \mathscr{H}: \ \psi \geqslant \varphi\}.$$

Then the indicator function $I_{\mathscr{K}_\varphi}$ of $\mathscr{K}_\varphi$ is given by

$$I_{\mathscr{K}_\varphi}(\psi) = \begin{cases} 0 & \text{if } \psi \in \mathscr{K}_\varphi, \\ +\infty & \text{if } \psi \notin \mathscr{K}_\varphi, \end{cases}$$

and the subgradient of the function $I_{\mathscr{K}_\varphi}$ is a multivalued mapping $\partial I_{\mathscr{K}_\varphi}$ defined on the domain $D(\partial I_{\mathscr{K}_\varphi}) = \mathscr{K}_\varphi$ by the formula (see [5])

$$\partial I_{\mathscr{K}_\varphi}(\psi) = \{\xi \in \mathscr{H}: \ \langle \xi, \eta - \psi \rangle \leqslant 0 \ \text{for all} \ \eta \in \mathscr{K}_\varphi\}, \tag{8}$$

$$\partial I_{\mathscr{K}_\varphi}(\psi) = \{\xi \in \mathscr{H}: \ \xi \leqslant 0, \ \text{and if} \ \psi(x) = \varphi(x), \ \text{then} \ \xi(x) = 0\}. \tag{9}$$

Our aim in this section is to show that under natural conditions the following evolutionary inclusion:

$$\frac{dV}{dt}(t, x) \in \mathscr{L} V(t, x) - \partial I_{\mathscr{K}_\varphi}(V(t, x)), \quad V(0, x) = \psi(x), \ x \in E, \ t \geqslant 0, \tag{10}$$

has a unique strong solution.

One says that a locally Lipschitz continuous $\mathscr{H}$-valued function $V(t)$, $t \geqslant 0$, is a *strong solution*, to the inclusion (10) if, for all $t \geqslant 0$, $V(t) \in \mathscr{K}_\varphi \cap \mathscr{D}(\mathscr{L})$, $V(0) = \psi$, and the inclusion (10) holds for almost all $t \geqslant 0$ and μ-almost all $x \in E$.

If $\varphi \in \mathscr{H}$, $\psi \in \mathscr{K}_\varphi$ and there exist functions $\psi_n \geqslant \varphi_n$, $n \in N$, converging in $\mathscr{H}$ to ψ and φ, respectively, for which the inclusions (10) have strong solutions V_n, $n \in N$, converging uniformly on bounded intervals of R^1_+ to a continuous functions V, then V is called a *weak solution* of (10). Weak solutions might be not unique. One gets the uniqueness imposing additional conditions on the approximating sequences (φ_n) and (ψ_n).

The following theorem is the main result of this section:

THEOREM 2. *Assume that* (A.1) *and* (A.2) *hold. If the functions* φ *and* ψ *are in* $\mathscr{D}(\mathscr{L})$ *and* $\psi \geqslant \varphi$, *then the inclusion* (10) *has a unique strong solution.*

Proof. The theorem will be a consequence of a result on maximal monotone operators (see e.g. [5]) and of a lemma.

Let $\mathscr{M}$ be a transformation from a set $D(\mathscr{M}) \subset \mathscr{H}$ into the set of non-empty subsets of a Hilbert space $\mathscr{H}$ and let ω be a real number. The transformation

$\mathcal{M}$ is said to be *ω-maximal monotone* if the operator $\mathcal{M}+\omega I$ is maximal monotone (see [5], p. 106, and [8], p. 82). If $\mathcal{M}$ is ω-maximal monotone, then, for arbitrary $\lambda \in (0, 1/\omega^+)$, the image of $I+\lambda\mathcal{M}$ is the whole $\mathcal{H}$ and, for arbitrary $y \in \mathcal{H}$, there exists a unique $x \in D(\mathcal{M})$ such that

$$y \in x+\lambda\mathcal{M}(x).$$

The unique element y is denoted by $\mathcal{J}_\lambda(x)$ and the family of transformations $\mathcal{J}_\lambda$, $\lambda \in (0, 1/\omega^+)$, is called the *resolvent* of $\mathcal{M}$. Operators $\mathcal{M}_\lambda = \lambda^{-1}(I-\mathcal{J}_\lambda)$, $\lambda \in (0, 1/\omega^+)$, are called *Yosida approximations* of $\mathcal{M}$. If $\mathcal{N}$ is a maximal monotone operator on $\mathcal{H}$, then the sum $\mathcal{M}+\mathcal{N}$ is ω-monotone but not always ω-maximal monotone.

The following result is due to Brézis et al. [6]:

THEOREM 3. *Assume that operators $\mathcal{M}$ and $\mathcal{N}$, defined on subsets of a Hilbert space $\mathcal{H}$, are ω-maximal monotone and maximal monotone, respectively. If for arbitrary $y \in \mathcal{H}$, arbitrary $\lambda > 0$ and arbitrary $\delta \in (0, 1/\omega^+)$ there exists a solution x_λ^δ of the problem*

$$y \in x+\delta(\mathcal{M}x+\mathcal{N}_\lambda x)$$

such that for each $\delta \in (0, 1/\omega^+)$ the functions $\mathcal{N}_\lambda x_\lambda^\delta$, $\lambda > 0$, are bounded as $\lambda \to 0$. Then the operator $\mathcal{M}+\mathcal{N}$, with the domain $D(\mathcal{M}) \cap D(\mathcal{N})$, is ω-maximal monotone.

First we derive from Theorem 3 the following crucial proposition:

PROPOSITION 1. *Assume that an operator $\mathcal{L}$ is an infinitesimal generator of a C_0-semigroup of positive linear operators (R_t) on a Hilbert space $\mathcal{H} = L^2(E, \mu)$ such that for some $\gamma \in R^1$*

$$|R_t\psi|_{\mathcal{H}} \leqslant e^{\gamma t}|\psi|_{\mathcal{H}}, \quad t \geqslant 0,\ \psi \in \mathcal{H}. \tag{11}$$

If $\varphi \in D(\mathcal{L})$, then the operator

$$-\mathcal{L}+\partial I_{\mathcal{K}_\varphi} \tag{12}$$

is γ-maximal monotone.

Proof. It is well known that a linear operator $-\mathcal{L}$ is γ-maximal monotone if and only if it generates a C_0-semigroup of linear operators satisfying (11). Let $\mathcal{M} = -\mathcal{L}$ and $\mathcal{N} = \partial I_{\mathcal{K}_\varphi}$. Then $\mathcal{N}$ is maximal monotone (see [5]) and

$$\mathcal{N}_\lambda(\eta) = -\lambda^{-1}(\varphi-\eta)^+, \quad \lambda > 0,\ \eta \in \mathcal{H}.$$

The proposition is now a consequence of Theorem 3 and of the following lemma. ∎

LEMMA 1. *If $\delta \in (0, 1/\gamma^+)$, $\lambda > 0$, $\psi \in \mathcal{H}$, then, for arbitrary $\psi \in \mathcal{H}$, the following equation:*

$$\psi = v-\delta\left(\mathcal{L}v+\lambda^{-1}(\varphi-v)^+\right) \tag{13}$$

has a unique solution v_λ *such that*

$$\|\lambda^{-1}(\varphi-v_\lambda)^+\|_{\mathscr{H}} \leqslant \|[\delta^{-1}(\varphi-\psi)-\mathscr{L}\varphi]^+\|_{\mathscr{H}}.$$

Proof. Define $\mathscr{R}_\sigma = (\sigma I-\mathscr{L})^{-1}$, $\sigma > \gamma^+$. Then the basic equation is equivalent to

$$v = \mathscr{R}_{1/\delta}\big(\delta^{-1}\psi+\lambda^{-1}(\varphi-v)^+\big)$$

or, by the resolvent identity, to

$$v = \mathscr{R}_{1/\delta+1/\lambda}(\delta^{-1}\psi)+\mathscr{R}_{1/\delta+1/\lambda}\big(\lambda^{-1}\big((\varphi-v)^+ +v\big)\big). \tag{14}$$

Since the norm of the operator $\mathscr{R}_{1/\delta+1/\lambda}$ is at most $(\delta^{-1}-\gamma^+\lambda^{-1})^{-1}$ and the real function $z\to\lambda^{-1}\big((a-z)^+ +z\big)$ is Lipschitz with constant λ^{-1}, the transformed equation (14) has a unique solution by the contraction mapping principle. Since $\varphi\in\mathscr{D}(\mathscr{L})$, there exists a function $\eta\in\mathscr{H}$ such that

$$\varphi = \mathscr{R}_{1/\delta}\eta, \qquad \delta^{-1}\varphi-\mathscr{L}\varphi = \eta,$$

or, again by the resolvent identity,

$$\varphi = \mathscr{R}_{1/\delta+1/\lambda}(\eta+\lambda^{-1}\varphi).$$

Since v_λ satisfies (14), we get, by subtraction,

$$v_\lambda-\varphi = \mathscr{R}_{1/\delta+1/\lambda}(\delta^{-1}\psi-\eta)+\lambda^{-1}\mathscr{R}_{1/\delta+1/\lambda}[(\varphi-v_\lambda)^+-(\varphi-v_\lambda)].$$

Consequently,

$$\varphi-v_\lambda \leqslant \mathscr{R}_{1/\delta+1/\lambda}(\eta-\delta^{-1}\psi)^+.$$

In particular,

$$(\varphi-v_\lambda)^+ \leqslant \mathscr{R}_{1/\delta+1/\lambda}(\eta-\delta^{-1}\psi)^+,$$

and

$$\begin{aligned}\|\lambda^{-1}(\varphi-v_\lambda)^+\| &\leqslant \lambda^{-1}\|\mathscr{R}_{1/\delta+1/\lambda}(\eta-\delta^{-1}\psi)^+\|_{\mathscr{H}}\\ &\leqslant \frac{\lambda^{-1}}{(\delta^{-1}-\omega)+\lambda^{-1}}\big\|\big(\delta^{-1}(\varphi-\psi)-\mathscr{L}\eta\big)^+\big\|_{\mathscr{H}}\\ &\leqslant \big\|\big(\delta^{-1}(\varphi-\psi)-\mathscr{L}\eta\big)^+\big\|_{\mathscr{H}},\end{aligned}$$

as required. ∎

To complete the proof of Theorem 2 it is enough to use Proposition 1 and recall (see [5]) that if an operator $\mathscr{M} = -\mathscr{L}+\partial I_{\mathscr{K}_\varphi}$ is γ-maximal monotone, then the differential inclusion

$$\frac{dz(t)}{dt}+\mathscr{M}z(t)\ni 0, \qquad z(0) = x\in\overline{D(\mathscr{M})}, \tag{15}$$

has a unique strong solution $z(t, x)$, $t\geqslant 0$, and for arbitrary $x\in\overline{D(\mathscr{M})}$ the inclusion (15) has a weak solution denoted also by $z(t, x)$, $t \geqslant 0$. Moreover, the operators $S(t)\colon \overline{D(\mathcal{M})} \to \overline{D(\mathcal{M})}$, $t \geqslant 0$, are given by

$$S(t)x = z(t, x), \quad t \geqslant 0, \ x \in \overline{D(\mathcal{M})}. \ ■$$

It is of interest to notice that we have implicitly shown the existence of a solution to the following stationary inclusion:

$$0 \in \mathcal{L}V(x) - \partial I_{\mathcal{K}_\varphi}(V(x)) \tag{16}$$

on the value function V for a stopping problem on an infinite time interval (see the Introduction). In fact, we have the following result:

THEOREM 4. *If the assumptions* (A.1) *and* (A.2) *are satisfied with* $a < -\omega$ *and the function* φ *is in* $\mathcal{D}(\mathcal{L})$, *then the inclusion* (16) *has a unique strong solution.*

Proof. Define $\mathcal{G} = \mathcal{L} - ((\omega + a)/2)I$. Then $\mathcal{G}$ generates a C_0-semigroup satisfying the inequality (11) with $\gamma = 0$. Consequently, the operator

$$-\mathcal{L} + \left(\frac{\omega + a}{2}\right) I + \partial I_{\mathcal{K}_\varphi}$$

is maximal monotone. In particular, for arbitrary $\lambda > 0$ the inclusion

$$0 \in \psi + \lambda\left(-\mathcal{L}\psi + \left(\frac{\omega + a}{2}\right)\psi\right) + \partial I_{\mathcal{K}_\varphi}(\psi) \tag{17}$$

has a unique solution $\psi \in D(\mathcal{L}) \cup \mathcal{K}_\varphi$. However, (17) is equivalent to

$$-\left(\frac{1}{\lambda} + \frac{\omega + a}{2}\right)\psi \in \mathcal{L}\psi - \partial I_{\mathcal{K}_\varphi}(\psi).$$

Taking $1/\lambda = -(\omega + a)/2$, we obtain the result. ■

3. WEAK SOLUTIONS AS VALUE FUNCTIONS

In this section we consider again the family of Markov processes $X(t, x)$, $t \geqslant 0$, $x \in E$, parametrized by the initial condition $x \in E$. Our main theorem identifies the value function as a weak solution to the Bellman inclusion. As in the Introduction, $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t)$ is an increasing family of σ-fields. The σ-fields $\mathcal{F}_t$, $t \geqslant 0$, are assumed to be complete, and the E-valued $(\mathcal{F}_t)$-adapted stochastic processes $X(\cdot, x)$ are Markovian with respect to a transition semigroup (P_t) and σ-fields $(\mathcal{F}_t)$ in the sense that

$$E\big(\psi(X(t+h, x)) \mid \mathcal{F}_t\big) = P_h \psi\big(X(t, x)\big) \ P\text{-a.s.} \tag{18}$$

for arbitrary $\psi \in B_b(E)$ and $t, h \geqslant 0$. Although the σ-fields $(\mathcal{F}_t)$ may be different for different processes, the semigroup (P_t) is fixed once for all.

Denote by $\boldsymbol{D}$ the space of all E-valued right-continuous functions having left-hand limits, defined on the interval $[0, +\infty)$, equipped with the Skorokhod topologies (see [4], Chapter 3, and [16]). Denote by $\mathcal{M}(E)$ the space of all probability measures on $(E, \mathscr{E})$ equipped with the topology of weak convergence. The spaces $\boldsymbol{D}$ and $\mathcal{M}(E)$ are also metric separable and complete. By P^x we denote the distribution of the process $X(\cdot, x)$ on $\boldsymbol{D}$. We will need the following assumption:

(A.3) *For arbitrary* $x \in E$,

$$\boldsymbol{P}\big(X(\cdot, x) \in \boldsymbol{D}\big) = 1$$

and the mapping $x \to P^x$ *from* E *to* $\mathcal{M}(\boldsymbol{D})$ *is continuous.*

The assumption (A.3) is equivalent to the continuous dependence of the laws of the Markov processes $X(\cdot, x)$, $x \in E$, on the initial condition and is satisfied in many interesting cases. It the space E is locally compact, it is sufficient to assume that, for arbitrary $\varphi \in C_0(E)$, $P_t\varphi(x)$, $t \geqslant 0$, $x \in E$, is continuous and, for each $t \geqslant 0$, $P_t\varphi(x) \in C_0(E)$. This condition is very close to our assumption (A.1). It is possible, however, to construct examples showing that the assumption (A.1) does not imply the continuity of the value function for continuous data (see [26]).

On the functions φ, ψ and on the (discount) function α we impose continuity and growth conditions, usually satisfied in applications.

(A.4) (i) *Functions* φ *and* ψ *are continuous and bounded on bounded sets and* $\varphi \leqslant \psi$.

(ii) *For an arbitrary compact set* $K \subset E$ *and arbitrary* $T > 0$:

$$\boldsymbol{E}\Big(\sup_{x \in K} \sup_{t \in [0,T]} \big(|\varphi(X(t, x))| + |\psi(X(t, x))|\big)\Big) < +\infty.$$

(iii) *For arbitrary* $T > 0$ *there exists a function* $\zeta \in \mathscr{H}$ *such that*

$$\boldsymbol{E}\Big(\sup_{t \in [0,T]} \big(|\varphi(X(t, x))| + |\psi(X(t, x))|\big)\Big) \leqslant \zeta(x), \qquad x \in E.$$

(iv) *The function* α *is continuous and bounded from above by a constant* a,

$$\alpha(x) \leqslant a, \qquad x \in E.$$

In this section, for a function g,

$$R_t g(x) = \boldsymbol{E}\Big(\exp\Big\{\int_0^t \alpha(X(s, x))\Big\}\, g(X(t, x))\Big), \qquad t \geqslant 0,\ x \in E.$$

We can now state the main result of the present section.

THEOREM 5. *Under the assumptions* (A.2)–(A.4) *the value function* $V(t, x)$, $t \geqslant 0$, $x \in E$, *is continuous and is a weak solution of the Bellman inclusion* (10). *If, in addition,* φ, $\psi \in D(\mathscr{L})$, *then* V *is the unique strong solution of* (10).

Proof. The fact that V is continuous has been established, in the present generality, in [29]. The proofs of the remaining parts of the theorem use the penalization technique (see [3] and [27]) and will be divided into several steps.

Step 1. *Functions φ and ψ are in $C_b(E)\cap D(\mathscr{L})$.*

Consider first the so-called *penalized Bellman equation*

$$\frac{dV^\lambda}{dt}(t,x) = \mathscr{L}V^\lambda(t,x)+\frac{1}{\lambda}(\varphi - V^\lambda(t,x))^+, \tag{19}$$

$$V^\lambda(0,x) = \psi(x), \qquad x\in E,\ t>0,$$

in its integral form

$$V^\lambda(t) = R_t\psi+\frac{1}{\lambda}\int_0^t R_s(\varphi - V^\lambda(t-s))^+ ds, \qquad t\geqslant 0. \tag{20}$$

Note that the transformation $\zeta\to(\varphi-\zeta)^+$ satisfies the Lipschitz condition both in $\mathscr{H}$ and in $C_b(E)$. By an easy contraction argument the equation (20) has a unique solution in $C([0,T],\mathscr{H})$, first for small T and then for all $T\geqslant 0$. By a similar argument the equation (20) has also a unique solution in $C([0,T],C_b(E))$ for arbitrary $T\geqslant 0$. It is clear that if $\varphi\in C_b(H)\cap\mathscr{H}$, then both solutions coincide.

Let $\lambda>0$ be an arbitrary positive number and $u(t)$, $t\geqslant 0$, an $(\mathscr{F}_t)$-adapted process taking values in the interval $[0,1/\lambda]$. Define functionals

$$\tilde{J}_s(u,x) = E\Big(\int_0^s\Big[\exp\Big\{\int_0^t\big(\alpha(X(\sigma,x))-u(\sigma)\big)d\sigma\Big\}u(t)\varphi(X(t,x))\Big]dt \tag{21}$$
$$+\exp\Big\{\int_0^s\big(\alpha(X(\sigma,x))-u(\sigma)\big)d\sigma\Big\}\psi(X(s,x))\Big)$$

and consider the following value function:

$$V_\lambda(s,x) = \sup_{0\leqslant u(\cdot)\leqslant 1/\lambda}\tilde{J}_s(u,x), \tag{22}$$

where the supremum is taken with respect to all processes $u(t)$, $t\geqslant 0$, $(\mathscr{F}_t)$-adapted, having values in the interval $[0,1/\lambda]$.

We need the following result:

PROPOSITION 2. *Assume that the conditions* (A2)–(A4) *hold and φ, $\psi\in C_b(E)$, $\psi\geqslant\varphi$. Then*

(i) $V^\lambda(s,x) = V_\lambda(s,x)$ *for all* $\lambda>0$, $s>0$, $x\in E$;

(ii) $V_\lambda(s,x)\uparrow V(s,x)$ *as* $\lambda\downarrow 0$ *for* $s>0$, $x\in E$.

Proof. We adapt the proof from [27] to the present, more general situation.

The following lemma is a generalization of the first part of Lemma 1 from [27] with a similar proof.

LEMMA 2. *Let $(x_t)_{t\geqslant 0}$, $(u_t)_{t\geqslant 0}$ and $(\alpha_t)_{t\geqslant 0}$ be progressively-measurable real processes such that x and u are bounded and α is bounded from above and locally integrable. If $T\in[0, +\infty]$ and $(w_t)_{t\in[0,t]}$ is a right-continuous process such that, for each $t\in[0, T]$,*

$$w_t = E\Big(\int_t^T \exp\Big\{\int_t^s \alpha_r\,dr\Big\}\, x_s\,ds \,|\, \mathscr{F}_t\Big)\ \ P\text{-a.e.},$$

then, for each $t\in[0, T]$,

$$w_t = E\Big(\int_t^T \exp\Big\{\int_t^s (\alpha_r-u_r)\,dr\Big\}(x_s+u_s W_s)\,ds \,|\, \mathscr{F}_t\Big)\ \ P\text{-a.e.}$$

Let $f(s, x) = (\varphi(x)-V^\lambda(s, x))^+$, $s\geqslant 0$, $x\in E$, where V^λ is a solution to the equation (21). Then, for a fixed $T>0$ and all $t\in[0, T]$, $x\in E$,

$$\int_t^T (R_{\sigma-t} f(T-\sigma, \cdot))(x)\,d\sigma = \lambda\big(V^\lambda(T-t, x)-R_{T-t}\psi(x)\big). \tag{23}$$

Define $x_t = f(T-t, X(t, x))$, $\alpha_t = \alpha(X(t, x))$, $u_t = u(t)$, and

$$w_t = E\Big(\int_t^T \exp\Big\{\int_t^s \alpha_r\,dr\Big\}\, x_s\,ds \,|\, \mathscr{F}_t\Big), \qquad t\in[0, T],\ x\in E.$$

By (23) we have

$$w_t = \lambda\big[\big(V^\lambda(T-t, X(t, x))\big)-R_{T-t}\psi(X(t, x))\big], \qquad t\in[0, T].$$

Applying the lemma with $t=0$ we obtain

$$\begin{aligned}\lambda[V^\lambda(T, x)-R_T\psi(x)] = E\Big(\int_0^T \exp\Big\{\int_0^s [\alpha(X(r, x))-u(r)]dr\Big\}\big[u(s)W_s \\ +\big(\varphi(X(s, x))-V^\lambda(T-s, X(s, x))\big)^+\big]\,ds\Big).\end{aligned} \tag{24}$$

However, by the Markov property,

$$\begin{aligned} &E\Big(\int_0^T \Big[\exp\Big\{\int_0^s \big(\alpha(X(T, x))-u(r)\big)dr\Big\}u(s)R_{T-s}\psi(X(s, x))\Big]\,ds\Big) \\ &= E\Big(\Big(\int_0^T \exp\Big\{-\int_0^s u(r)\,dr\Big\}\Big)\exp\Big\{\int_0^T \alpha(X(r, z))\,dr\Big\}\psi(X(T, x))\Big) \\ &= -E\Big(\exp\Big\{\int_0^T \big(\alpha(X(r, x))-u(r)\big)dr\Big\}\psi(X(T, x))\Big)+R_T\psi(x).\end{aligned} \tag{25}$$

Consequently, by (24) and (25),

$$\begin{aligned}&\lambda[V^\lambda(T, x)-R_T\psi(x)] \\ &= \lambda E\Big(\exp\Big\{\int_0^T \big(\alpha(X(r, x))-u(r)\big)dr\Big\}\psi(X(T, x))\Big)-\lambda R_T\psi(x)\end{aligned}$$

$$+E\Big(\int_0^T \exp\Big\{\int_0^s \big(\alpha(X(r, x)) - u(r)\big)dr\Big\}\Big\{\lambda u(s)\varphi(X(s, x))$$
$$-\lambda u(s)\big[\varphi(X(s, x)) - V^\lambda(T-s, X(s, x))\big]$$
$$+\big(\varphi(X(s, x)) - V^\lambda(T-s, X(s, x))\big)^+\Big\}\, ds\Big).$$

Therefore,

$$V^\lambda(T, x) = \hat{J}_T(u, x) + E\left\{\int_0^T\left[\frac{1}{\lambda}\big(\varphi(X(s, x)) - V^\lambda(T-s, X(s, x))\big)^+\right.\right.$$
$$\left.\left. -u(s)\big(\varphi(X(s, x)) - V^\lambda(T-s, X(s, x))\big)\right] ds\right\}.$$

If $u(s) \in [0, 1/\lambda]$, $s \in [0, T]$, then $V^\lambda(T, x) \geqslant \hat{J}_T(u, x)$. Moreover, if

$$\hat{u}(s) = \begin{cases} 1/\lambda & \text{if } \varphi(X(s, x)) \geqslant V^\lambda(T-s, X(s, x)), \\ 0 & \text{if } \varphi(X(s, x)) < V^\lambda(T-s, X(s, x)), \end{cases}$$

then $V^\lambda(T, x) = \hat{J}_\lambda(\hat{u}, x)$, so part (i) of the proposition holds true.

We pass to the proof of (ii). By part (i), the function $V_\lambda(s, x)$, $\lambda > 0$, is decreasing in $\lambda > 0$. To show that $V(T, x) \geqslant V_\lambda(T, x)$ it is convenient to introduce a new probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \boldsymbol{P})$ with

$$\hat{\Omega} = \Omega \times [0, +\infty], \qquad d\hat{\boldsymbol{P}}(\omega, \lambda) = \exp\Big\{-\int_0^\lambda u(s, \omega)\, ds\Big\}\, u(\lambda, \omega)\, d\lambda\, dP(\omega),$$
$$\hat{\mathscr{F}} = \mathscr{F} \otimes B[0, +\infty],$$

where $B[0, t]$, $t \in [0, +\infty]$, denotes the σ-field of subsets of $[0, +\infty]$, generated by Borel subsets of $[0, t]$. Let, in addition,

$$\hat{X}(t, x, \hat{\omega}) = X(t, x, \omega) \quad \text{ for } \hat{\omega} = (\lambda, \omega) \in \hat{\Omega}.$$

Then $\hat{X}$ is a Markov process with respect to σ-fields $(\mathscr{F}_t \otimes B[0, t])$ and with respect to the initial transition semigroup (P_t). Moreover, a random variable $\hat{\tau}\colon \hat{\Omega} \to [0, +\infty]$, defined as

$$\hat{\tau}(\lambda, \omega) = \lambda, \qquad \omega \in \Omega,\ \lambda \in (0, +\infty],$$

is an $(\hat{\mathscr{F}}_t)$-stopping time. Since

$$\tilde{J}_T(u, x) = \hat{E}\Big(\exp\Big\{\int_0^{\hat{\tau}} \alpha(\hat{X}(\sigma, x))\, d\sigma\Big\}\big[\varphi(\hat{X}(\hat{\tau}, x))\chi_{\hat{\tau} < T} + \psi(X(\hat{\tau}, x))\chi_{\hat{\tau} = T}\big]\Big),$$

we have $V(T, x) \geqslant V_\lambda(T, x)$.

Let, finally, τ be a stopping time with only a finite number of values $t_1 < t_2 < \ldots < t_N = T$. If natural n is such that $t_j + 1/n < t_{j+1}$ for

$j = 1, 2, \ldots, N-1$, define

$$u_n(s) = n^2 \sum_{j=1}^{N-1} \chi(s)_{[t_j, t_j+1/n]}(s), \qquad s \in [0, T].$$

Since functions φ and ψ are continuous and bounded and the process X is right-continuous, we obtain

$$J_T(\tau, x) = \lim_{n \to +\infty} \tilde{J}_T(u_n, x).$$

This and the fact that $V(T, x) \geqslant V_\lambda(T, x)$ imply easily (ii). ∎

We go back to the proof of Theorem 5 and define $\mathcal{M} = -\mathcal{L}$ and $\mathcal{N} = \partial I_{\mathcal{K}_\varphi}$. Then the Yosida approximations $\mathcal{N}_\lambda$ are given by the formula

$$N_\lambda(\eta) = \lambda^{-1}(\varphi - \eta)^+, \qquad \lambda > 0, \; \eta \in \mathcal{H}.$$

Moreover, the penalized equations (19) are of the form

$$dV^\lambda/dt + \mathcal{M}V^\lambda + \mathcal{N}_\lambda V^\lambda = 0, \qquad V^\lambda(0) = \psi(x), \; x \in E, \tag{26}$$

and the solutions $V^\lambda(t)$, $t \geqslant 0$, form a continuous semigroup of transformations $\psi \to S^\lambda(t)\psi$, $t \geqslant 0$, on $\mathcal{H}$. By Theorems 1 and 2, the operator $\mathcal{M} + \mathcal{N}$ is $((\omega + a)/2)$-maximal monotone and the value function V is identical with the strong solution of the equation

$$-dV/dt \in \mathcal{M}V + \mathcal{N}V, \qquad V(0) = \psi(x), \; x \in E. \tag{27}$$

Let $S(t)$, $t \geqslant 0$, be the semigroup determined by (27). We need a version of Benilan's theorem [1]:

THEOREM 6. *Assume that the assumptions of Theorem 2 hold. Then, for arbitrary* $x \in \overline{D(\mathcal{M} + \mathcal{N})}$, $S^\lambda(t)x \to S(t)x$ *uniformly on bounded subsets of* $[0, +\infty)$.

Proof. One shows (see [5], p. 35) that for arbitrary $\delta \in (0, 1/\omega^+)$ the limit $x_\infty^\delta = \lim_{\lambda \to 0} x_\lambda^\delta$ is the unique solution of the inclusion

$$y \in x + \delta\big(\mathcal{M}(x) + \mathcal{N}(x)\big).$$

This means that, for arbitrary $\delta \in (0, 1/\omega^+)$ and $y \in \mathcal{H}$,

$$\big(I + \delta(\mathcal{M} + \mathcal{N}_\lambda)\big)^{-1} y \to \big(I + \delta(\mathcal{M} + \mathcal{N})\big)^{-1} y \qquad \text{as } \lambda \to 0.$$

The result follows now from Benilan's theorem (see Theorem 4.2 in [5] or [1]). ∎

From Lemma 1 and Benilan's theorem it follows that $S^\lambda \psi \to S\psi$ uniformly on bounded intervals of R_+^1, as functions with values in $\mathcal{H}$. Since $S^\lambda \psi = V_\lambda$

and V_λ converges to the value function V pointwise, V is the strong solution of Bellman's inclusion (10).

In the remaining part of the proof all elements from $\mathscr{H}$ having continuous versions are identified with those versions.

Step 2. *Functions φ and ψ are in $C_b(E)$.*

For natural $n > \frac{1}{2}\omega + a$, define

$$\mathscr{G}_n \eta = \int_0^{+\infty} e^{-ns} R_s \eta ds, \qquad \eta \in \mathscr{H}.$$

The functions

$$\varphi_n = n\mathscr{G}_n \varphi, \qquad \psi_n = n\mathscr{G}_n \psi, \qquad n > (\omega + a)/2,$$

are in $D(\mathscr{L})$ and $\varphi_n \to \varphi$, $\psi_n \to \psi$ as $n \to +\infty$ both in $\mathscr{H}$ and uniformly on compact subsets of E. However, under the continuity condition the following *compact confinement property* holds (see [29]):

For an arbitrary compact set $K \subset E$, $T > 0$ and $\varepsilon > 0$ there exists a compact set $L \subset E$ such that

$$P\big(X(t, x) \in L \textit{ for all } t \in [0, T]\big) \geqslant 1 - \varepsilon \quad \textit{for all } x \in K.$$

Therefore, the corresponding continuous functions (V_n) converge to the value function V uniformly on compact subsets of $[0, +\infty) \times E$. This in turn implies that V_n, regarded as $\mathscr{H}$-valued functions, converge uniformly on bounded intervals of R^1_+ to V.

Step 3. *Functions φ and ψ satisfy the assumption* (A.4).

By Step 2 it is enough to show that there exist sequences (φ_n) and (ψ_n) of functions from $C_b(E)$ such that $\varphi_n \to \varphi$ and $\psi_n \to \psi$ as $n \to +\infty$ in $\mathscr{H}$, $\varphi_n \leqslant \psi_n$, and the corresponding value functions V_n, $n \in N$, converge to V, as $\mathscr{H}$-valued functions, uniformly on bounded intervals. Moreover, V is the value function corresponding to the data (φ, ψ).

Choose φ_n, ψ_n, $n \in N$, identical to φ, ψ, on balls B_n with a fixed center x_0 and radius n and such that on E:

$$|\varphi_n| \leqslant |\varphi|, \qquad |\psi_n| \leqslant |\psi|, \qquad n \in N.$$

We show first that the corresponding value functions V_n, $n \in N$, converge uniformly on compact subsets of $[0, +\infty) \times E$ to the value function corresponding to φ, ψ. Let T be a fixed positive number, K a fixed compact subset of E, and (L_n) an increasing sequence of compact sets (L_n) such that, for all $x \in K$,

$$P\big(X(t, x) \in L_n \text{ for all } t \in [0, T]\big) \geqslant 1 - 1/n, \qquad n = 1, 2, \ldots$$

Without any loss of generality one can assume that $L_n \subset B_n$. Functions V_n are continuous by Step 2 and

$$V_n(s, x) = \sup_{\tau \leqslant s} E\Big(\exp\Big\{\int_0^\tau \alpha_n(X(\sigma, x))\,d\sigma\Big\}\big[\varphi_n(X(\tau, x))\chi_{\tau<s} + \psi_n(X(s, x))\chi_{\tau=s}\big]\Big).$$

Define a random variable

$$\xi = \sup_{x \in K} \sup_{t \in [0,T]} \big|\psi(X(t, x))\big|,$$

and events

$$A_n = \{X(\sigma, x) \in L_n^c \text{ for some } \sigma \in [0, T]\}.$$

Note that for a constant C and for arbitrary $s \in [0, T]$, $x \in K$:

$$|V_n(s, x) - V(s, x)| \leqslant C \sup_{\tau \leqslant s} E\big(|\psi(X(\tau, x))|\,\chi_{A_n}\big) \leqslant CE(\xi\chi_{A_n}).$$

By the assumption (A.4), ξ is an integrable random variable and, by the continuity condition (see Step 2), $P(A_n) \to 0$ as $n \to \infty$. Consequently, $E(\xi\chi_{A_n}) \to 0$ as $n \to \infty$ and

$$\sup_{x \in K} \sup_{t \in [0,T]} |V_n(s, x) - V(s, x)r| \to 0 \quad \text{as} \to +\infty.$$

This proves the uniform convergence on compact subsets of $[0, +\infty) \times E$. To complete the proof note that

$$|V_n(s, x) - V(s, x)| \leqslant \zeta(x), \quad s \in [0, T],\ x \in E.$$

Fix $\varepsilon > 0$ and let $L \subset E$ be a compact set and n_0 a natural number such that, for $n \geqslant n_0$, $s \in [0, T]$, $x \in L$:

$$\int_{L^c} \zeta^2(x)\,\mu(dx) \leqslant \varepsilon, \quad |V_n(s, x) - V(s, x)| \leqslant \varepsilon.$$

Then

$$\sup_{s \in [0,T]} \int_E |V_n(s, x) - V(s, x)|^2\,\mu(dx) \leqslant \varepsilon^2(4 + \mu(L)).$$

This completes the proof. ∎

4. APPLICATIONS

4.1. American options in finite market. Assume that a financial market consists of d random assets with prices $Y(t) = (Y_1(t), \ldots, Y_d(t))$, $t \geqslant 0$, satisfying equations of the form

$$dY_k(t) = Y_k(t-)\,d\zeta_k(t), \tag{28}$$

$$Y_k(0) = y_k > 0, \quad k = 1, 2, \ldots, d,\ t \geqslant 0, \tag{29}$$

where the process $\zeta(t)$, $t \geqslant 0$, with the components $\zeta_1(t), \ldots, \zeta_d(t)$, is a general, homogeneous in time process with independent increments on R^d with the drift vector, the covariance matrix and the jump measure denoted by $\tilde{a}$, $\tilde{Q}$ and $\tilde{\nu}$, respectively. We will assume that the jump measure $\tilde{\nu}$ is concentrated on $(-1, +\infty)^d$ to avoid negative prices. The solution $Y(t) = Y(t, y)$, $t \geqslant 0$, $y \in R^d_+$, is then given by the Doleans–Dade formula

$$Y(t, y) = e^{X(t,x)}, \qquad t \geqslant 0,$$

where

$$X(t, x) = x + Z(t), \qquad x = \ln y,$$

and $Z(t)$, $t \geqslant 0$, is a new, homogeneous in time process with independent increments on R^d, with parameters a, Q and ν. Explicit relations between parameters characterizing ζ and Z can easily be written down. The following result was proved in [30] (see also [31]).

PROPOSITION 3. (i) *If* $r > d$, $\kappa > 0$, *and*

$$\int_{|y|>1} |y|^r \nu(dy) < +\infty, \tag{30}$$

then the measures μ *and* $\tilde{\mu}$ *on* R^d *and* R^d_+,

$$\mu(dx) = \frac{1}{1+\kappa|x|^r} dx, \qquad \tilde{\mu}(dy) = \frac{1}{1+\kappa|\ln y|^r}\left(\prod_{k=1}^{d} \frac{1}{y_k}\right) dy,$$

are ω*-excessive for processes* X *and* Y, *respectively, for sufficiently large* ω.

(ii) *If for some* $\gamma > 0$

$$\int_{|y|>1} e^{\gamma|y|} \nu(dy) < +\infty, \tag{31}$$

then the measures μ *and* $\tilde{\mu}$ *on* R^d *and* R^d_+,

$$\mu(dx) = e^{-\gamma|x|} dx, \qquad \tilde{\mu}(dy) = e^{-\gamma|\ln y|}\left(\prod_{k=1}^{d} \frac{1}{y_k}\right) dy,$$

are ω*-excessive for processes* X *and* Y, *respectively, for sufficiently large* ω.

It follows from the above propositions that the general theory, developed in the previous sections, can be applied to the price process Y. To be more specific let us recall that real functions $\tilde{\varphi}$ defined on R^d_+ are called, in finance, *contingent claims* and one of the objects of the financial theory is to find the rational price $\tilde{V}$ of $\tilde{\varphi}$ defined by the formula

$$\tilde{V}(t, y) = \sup_{\tau \leqslant t} E\left(e^{-R\tau} \tilde{\varphi}(Y(\tau, y))\right), \qquad t \geqslant 0, \ y \in R^d_+.$$

The function $\tilde{V}$ is called the *upper price* of an American option and the number R stands for the *interest rate constant.* If $d = 1$, κ is a positive constant and $\tilde{\varphi}(y) = (y-K)^+$ or $\tilde{\varphi}(y) = (K-y)^+$, $y > 0$, the corresponding options are called *call* and *put options,* respectively. If $d = 2$, b is a positive constant, $\tilde{\varphi}(y_1, y_2) = (y_1 - by_2)^+$, $y_1, y_2 > 0$, the option is called the *Margrabe option.* If d is arbitrary, $b_1, \ldots, b_d \geqslant 0$ and $\tilde{\varphi}$ is either

$$\tilde{\varphi}(y_1, \ldots, y_d) = (b_1 y_1 + \ldots + b_d y_d - K)^+$$

or

$$\tilde{\varphi}(y_1, \ldots, y_d) = \big(K - (b_1 y_1 + \ldots + b_d y_d)\big)^+,$$

then the options are called *basket call* and *basket put options* (see [23]).

Instead of characterizing the functions $\tilde{V}$ as solutions to appropriate Bellman's inequalities in the spaces $L^2(R_d^+, \tilde{\mu})$, where $\tilde{\mu}$ are excessive measures from Proposition 3, it is equivalent to study the function

$$V(t, x) = \sup_{\tau \leqslant t} E\big(e^{-R\tau} \varphi(X(t, x))\big) = \tilde{V}(t, e^x), \qquad x \in R^d, \tag{32}$$

where

$$\varphi(x) = \tilde{\varphi}(e^x), \qquad x \in R_+^d.$$

Denote by (P_t) the transition semigroup on $L^2(R^d, \mu)$ corresponding to X. Its generator $\mathscr{A}$ is given, on twice continuously differentiable functions φ, by the formula

$$\begin{aligned} A\varphi(x) &= \langle a, D\varphi(x)\rangle + \frac{1}{2}\operatorname{tr} QD^2 \varphi(x) \\ &+ \int_{|y| \leqslant 1} \big(\varphi(x+y) - \varphi(x) - \langle D\varphi(x), y\rangle\big)\, \nu(dy) \\ &+ \int_{|y| > 1} \big(\varphi(x+y) - \varphi(x)\big)\, \nu(dy), \qquad x \in R^d. \end{aligned} \tag{33}$$

We introduce the following assumption:

(A.5) *The function φ is twice continuously differentiable and, for $k, l = 1, \ldots, d$,*

$$\varphi, \frac{\partial \varphi}{\partial x_k}, \frac{\partial^2 \varphi}{x_k \partial x_l} \in L^2(R^d, \mu).$$

Using Itô's lemma it is possible to show that if μ is ω-excessive for (P_t) and (A.5) holds, then φ is in the domain of the generators A. The corresponding Bellman inclusion can be written in the form

$$\begin{aligned} &\frac{\partial V}{\partial t}(t, x) \in AV(t, x) - RV(t, x) - \partial I_{\mathscr{K}_\varphi}\big(V(t, x)\big), \\ &V(0, x) = \varphi(x), \qquad x \in R^d,\ t \geqslant 0. \end{aligned} \tag{34}$$

THEOREM 7. *Assume that, for some natural $r > d$,*

$$\int_{R^d} |x|^r \nu(dx) < +\infty.$$

(i) *If φ is a continuous and bounded function, then the function V, given by* (32), *is a weak solution in $L^2(R^d, 1/(1+|x|^r))$ of the inequality* (34).

(ii) *If, in addition, φ satisfies* (A.5), *then V is the unique strong solution of* (34).

Proof. We apply Theorems 4 and 5. The assumption (A.1) is trivially satisfied because $\alpha \equiv -R$. By Proposition 3 the measure $\mu(dx) = (1+|x|^r)^{-1} dx$ is ω-excessive for sufficiently large ω. Thus (A.2) holds. Since $X(t, x) = x+Z(t)$, $t \geqslant 0$, $x \in R^d$, where Z has trajectories in $D(0, +\infty)$, the condition (A.3) holds as well. Moreover, in the present situation $\psi = \varphi$, so if φ is bounded, the assumption (A.4) is also satisfied and the proof of part (i) of the theorem is complete. If in addition φ satisfies (A.5), then $\varphi \in D(\mathscr{A})$ and, therefore, by the second part of Theorem 4, the part (ii) holds as well. ∎

THEOREM 8. *Assume that*

$$E\left(\exp\{\sup_{t\leqslant 1} |Z(t)|\}\right) < +\infty \quad \textit{and} \quad \int_{R^d} e^{\gamma|x|} \nu(dx) < +\infty$$

for some $\gamma > 2$.

(i) *If φ is a continuous function such that, for a constant $c > 0$,*

$$|\varphi(x)| \leqslant c(1+e^{|x|}), \qquad x \in R^d,$$

then V given by (32) *is a weak solution in $L^2(R^d, e^{\gamma|x|} dx)$ of the inequality* (34).

(ii) *If, in addition, φ satisfies* (A.5), *then V is the unique strong solution of* (34).

Proof. We proceed in the same way as in the proof of the previous theorem. We check, for instance, that (A.4) holds. Note that, by our assumptions, for arbitrary $r > 0$ and T,

$$E\left(\sup_{|x|\leqslant r} \sup_{t\leqslant T} |\varphi(X(t, x))|\right) < +\infty,$$

so (A.4) (i) holds. Since, for a constant c_1,

$$E\left(\sup_{t\leqslant T} |\varphi(X(t, x))|\right) \leqslant c+c_1 e^{|x|}, \qquad x \in R^d,$$

and

$$\int_{R^d} e^{2|x|} e^{-\gamma|x|} dx < +\infty,$$

the assumption (A.4) (ii) holds as well. This way the proof is complete. ∎

Remark 1. The condition $\gamma > 2$ must hold if functions like $\varphi(x) = (e^x - K)^+$, $x \in R^1$, are to be covered by the theorem. Results similar to those of Theorem 8 have been obtained earlier by Zhang [32] and Mastroeni and Matzeu [20] by different methods and under the additional assumption that the jump measure ν was finite.

4.2. American options in infinite market. Denote by $\mathscr{P}(t, \zeta)$, $t \geqslant 0$, $\zeta > 0$, the price at moment t of a bond expiring at moment $t+\zeta$, and by $X(t, \zeta)$, $t \geqslant 0$, $\zeta > 0$, the so-called forward rate function related to $\mathscr{P}$ by the formula

$$\mathscr{P}(t, \zeta) = \exp\{-\int_0^\zeta X(t, \eta)\, d\eta\}.$$

Equivalently,

$$X(t, \zeta) = -\frac{\partial}{\partial \zeta} \ln \mathscr{P}(t, \zeta), \quad t \geqslant 0,\ \zeta > 0.$$

Following Heath et al. [14] and Musiela [22] we assume that the process X is a solution of an evolution equation of the form

$$\text{(35)} \qquad dX(t, \zeta) = \left[\frac{\partial}{\partial \zeta} X(t, \zeta) + \sigma(\zeta) \int_0^\zeta \sigma(\eta)\, d\eta\right] dt + \sigma(\zeta)\, dW(t),$$

$$X(0, \zeta) = x(\zeta), \quad \zeta \geqslant 0,\ t \geqslant 0,$$

where $W(t)$, $t \geqslant 0$, is a real Wiener process defined on a probability space $(\Omega, \mathscr{F}, P)$. The real-valued function σ is the so-called *volatility function* of the bond market. The solution to (35) with the initial function x will be denoted by $X(t, x, \zeta)$ or, shortly, $X(t, x)$. It is mathematically convenient and economically meaningful to consider the equation (35) in the space $H^1 = H^1([0, +\infty))$ of all absolutely continuous functions $x: R^1_+ \to R^1$ such that

$$\|x\|_1 = \left(|x(0)|^2 \int_0^{+\infty} |x'(\zeta)|^2\, d\zeta\right)^{1/2} < +\infty.$$

The set H^1 equipped with the norm $\|\cdot\|_1$ is a Hilbert space. Let $S(t)$, $t \geqslant 0$, be the left-shift semigroup on H^1 given by the formula

$$S(t)\, x(\zeta) = x(t+\zeta), \quad t \geqslant 0,\ \zeta \geqslant 0,\ x \in H^1.$$

The generator A of S is the first derivative operator,

$$Ax(\zeta) = \frac{d}{d\zeta} x(\zeta), \quad \zeta \geqslant 0,$$

with the domain $D(A) = H^2([0, +\infty))$ consisting of all functions x such that $x \in H^1$, x' is absolutely continuous, and $x'' \in L^2(0, +\infty)$. The state equation (35) can be written in the form

$$(36) \qquad dX = (AX + a) + \sigma dW, \qquad X(0) = x,$$

where $a(\zeta) = \sigma(\zeta)\int_0^\zeta \sigma(\eta)\,d\eta$, $\zeta \geqslant 0$, and W is a real-valued standard Wiener process ($U = R^1$, $Q = 1$).

If functions σ and a belong to H^1, then the equation (35) has a unique solution X. The solution is a generalized Ornstein–Uhlenbeck process discussed in an earlier section. If $\mathscr{A}$ is the generator of the transition semigroup (P_t) extended to $L^2(H^1, \mu)$, where μ is an ω-excessive measure, then, for $x \in D(A)$ and smooth φ: $H^1 \to R^1$,

$$\mathscr{A}\varphi(x) = \tfrac{1}{2}\langle D_{xx}^2 \varphi(x)\sigma, \sigma\rangle + \langle D_x \varphi(x), Ax + a\rangle.$$

Fix now two increasing sequences of positive numbers $T_1 < T_2 < \ldots < T_m < T$ and $b_1, b_2, \ldots, b_m$ and define

$$(37) \qquad \varphi(x) = \Big(1 - \sum_{k=1}^{m} b_k \exp\Big\{\int_0^{T_k} x(\zeta)\,d\zeta\Big\}\Big)^+, \qquad x \in H^1.$$

The rational price of the American payer swaption (see Gątarek and Musiela [11]) is a function $V(t, x)$, $t \in [0, T]$, $x \in H^1$, given by the formula

$$(38) \qquad V(t, x) = \sup_{\tau \leqslant t} E\Big(\exp\Big\{-\int_0^{\tau} X^+(s, x, 0)\,ds\Big\}\,\varphi\big(X(\tau, x)\big)\Big).$$

Compared with Gątarek and Musiela (see [11]) we use the positive process $X^+(s, x, 0)$, $s \geqslant 0$, instead of $X(s, x, 0)$, $s \geqslant 0$. This is done also in economical literature to avoid negative rates. Taking into account Theorems 4 and 5 we arrive at the following result:

THEOREM 9. *Assume that $a, \sigma \in H^1$. Then the swaption price V given by* (38) *is a weak solution of the following inclusion*:

$$(39) \qquad \frac{\partial V}{\partial t}(t, x) \in \mathscr{A}V(t, x) - x^+(0)\,V(t, x) - \partial I_{\mathscr{K}_\varphi}\big(V(t, x)\big),$$

$$V(0, x) = \varphi(x), \qquad x \in H^1.$$

Replacing φ by its smooth approximation we see that the corresponding V is the unique strong solution of (39).

Remark 2. Using the concept of viscosity solutions Gątarek and Święch proposed in [12] an alternative characterization of the swaption price.

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